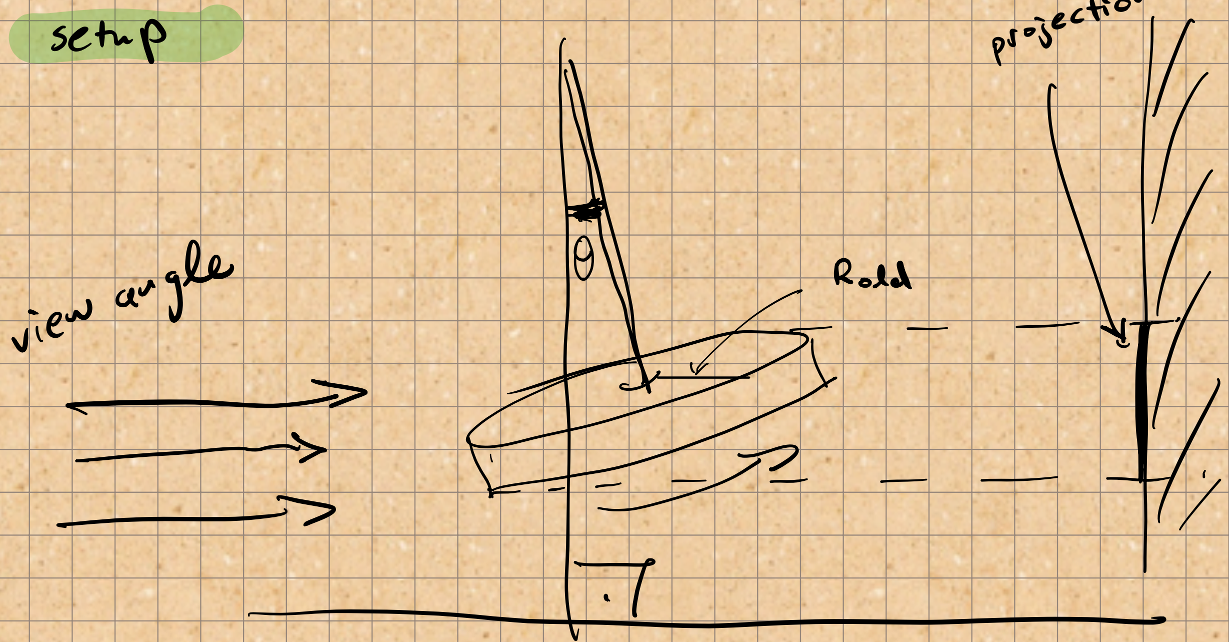
27 May 2020

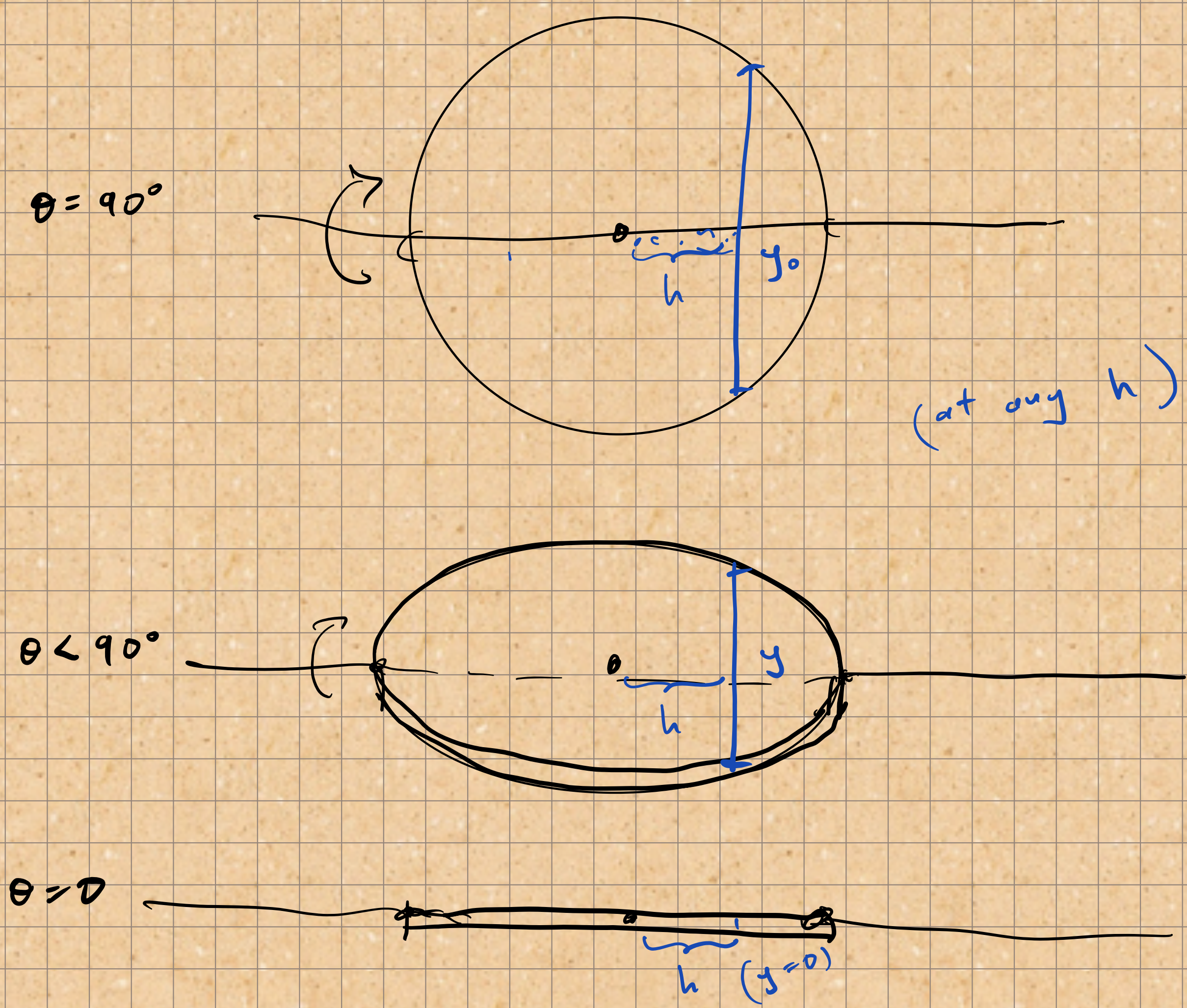
Increasing cutter radius from R_{old} to R_{new} across a width w by increasing θ

setup



Q: what θ corresponds to given projected radius, R_{new}, given R_{old}?

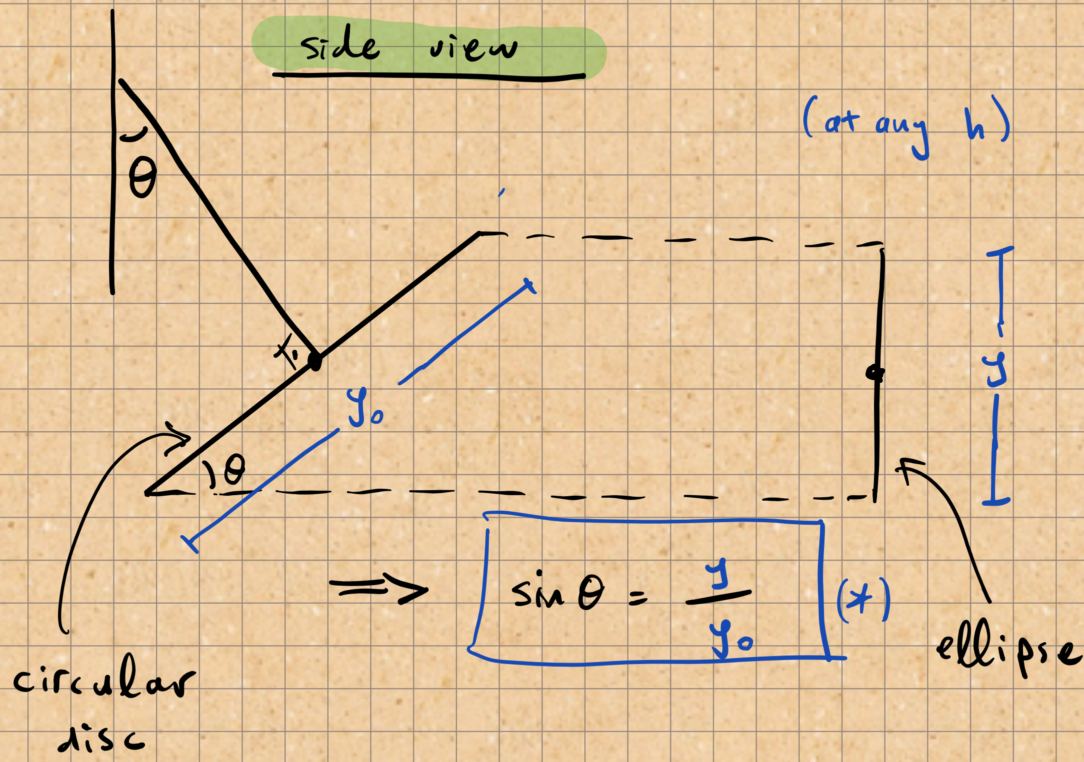
draw projection



(eventually: $2 \times h_{max}$ = width of neck = w)

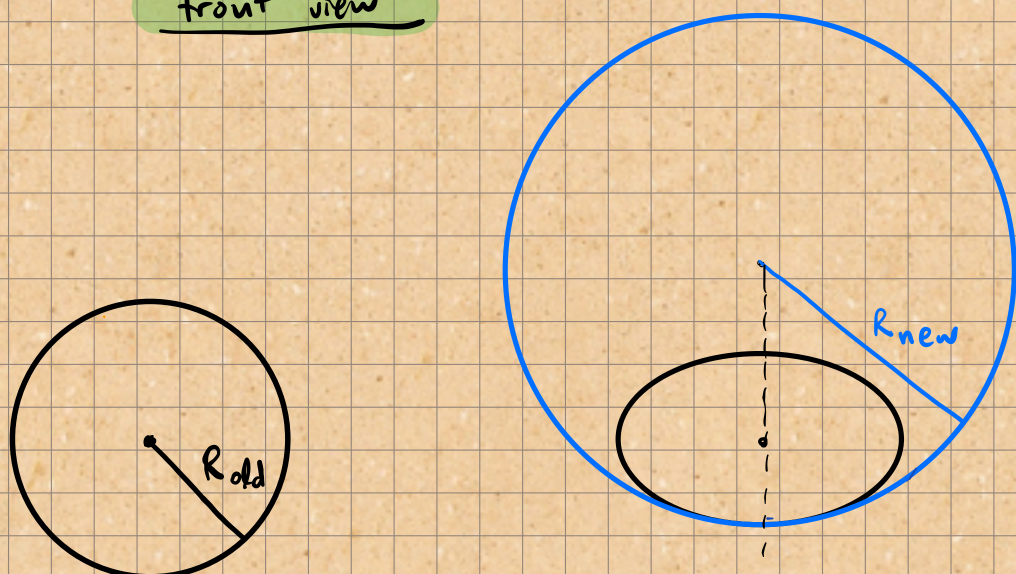
side view

(at any h)



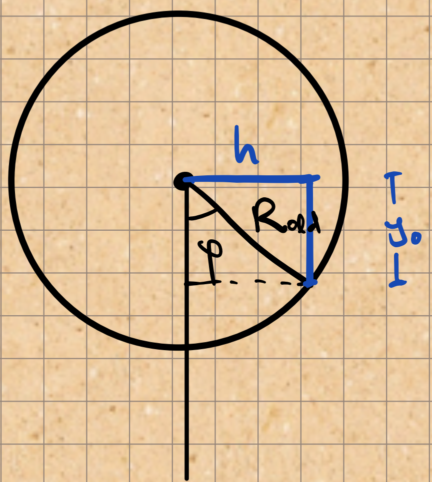
$$\Rightarrow \quad \boxed{\sin\theta = \frac{y}{y_0}} \quad (*)$$

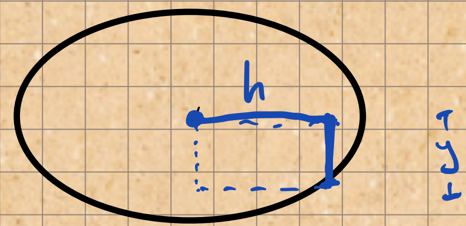
front view



$\theta = 90°$ $\qquad$ $\theta < 90°$

Question : $\theta(R_{new}, R_{old}) = ?$





$$\boxed{\begin{aligned} \sin\theta &= y/y_0 \quad \longleftarrow \text{from } (*) \\ y_0 &= R_{old}\cos\varphi \quad \longleftarrow \text{trig. identity} \end{aligned}}$$

- Will compute new radius, R_{new}, by using that it is determined by rate of change of y w.r.t. h.:

from boxed eqns above & trig:

$$\boxed{y = R_{old}\sin\theta\cos\varphi}$$

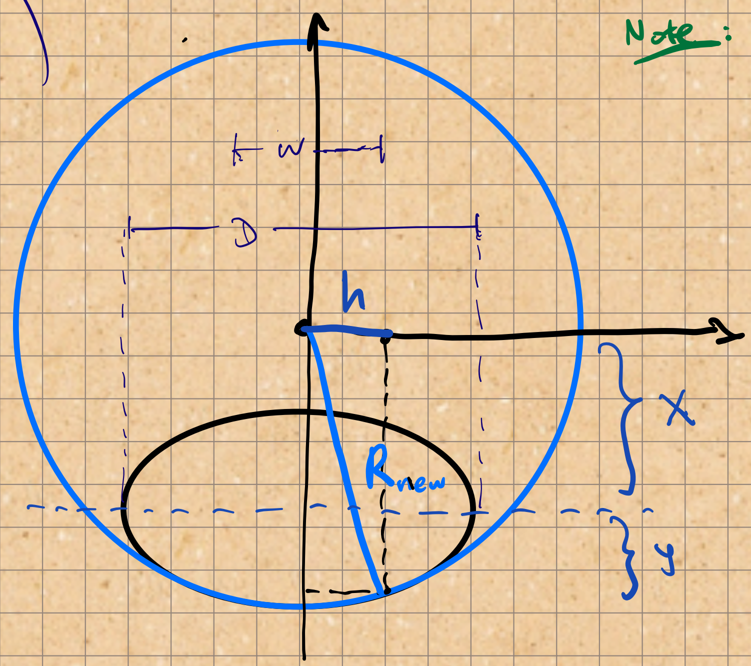
$$\boxed{h = R_{old}\sin\varphi}$$

variations:
$$\left.\begin{cases} \delta h = R_{old}\cos\varphi\ \delta\varphi \\ \delta y = -R_{old}\sin\theta\ \sin\varphi\ \delta\varphi \end{cases}\right\} \Rightarrow$$

$$\delta y = -\cancel{R_{old}}\sin\theta\ \sin\varphi\ \frac{\delta h}{\cancel{R}\cos\varphi} \Rightarrow$$

$$\boxed{\frac{dy}{dh} = -\sin\theta\ \tan\varphi}$$

(assume $D \gg W$)



Note: The radius R_{new} changes as we change h because the ellipse curvature changes.

for "some" x (we don't need to know x)

$$R_{new}^2 = h^2 + (x+y)^2 \qquad \text{(Pythagoras)}$$

$$\Rightarrow \quad 0 = 2h\,\delta h + 2(x+y)\,\delta y \qquad \text{(variation keeping } R_{new} \text{ \& } x \text{ fixed)}$$

$$\Rightarrow \quad 0 = h\,\delta h + \sqrt{R_{new}^2 - h^2}\;\delta y$$

$$\Rightarrow \quad \boxed{\frac{dy}{dh} = -\frac{h}{\sqrt{R_{new}^2 - h^2}}}$$

defining equation of R_{new} at h

But: $\dfrac{dy}{dh} = -\sin\theta \tan\varphi$ & $h = R_{old}\sin\varphi$

$$\Rightarrow \quad -\frac{(R_{old}\sin\varphi)}{\sqrt{R_{new}^2 - (R_{old}\sin\varphi)^2}} = -\sin\theta\tan\varphi$$

(setting 2 derivatives equal)

Solve for $\sin\theta$ & rearrange:

(recall $\tan\varphi = \sin\varphi/\cos\varphi$)

$$\Rightarrow \boxed{\sin\theta = \left(\frac{\cos\varphi}{\sqrt{1-\left(\frac{R_{old}}{R_{new}}\right)^2 \sin^2\varphi}}\right)\left(\frac{R_{old}}{R_{new}}\right)}$$

exact

at φ

In terms of h:

Recall: $h = R_{old}\sin\varphi \Rightarrow$

$$\sin\varphi = \frac{h}{R_{old}} \quad \& \quad \cos\varphi = \cos\left\{\sin^{-1}\left(\frac{h}{R_{old}}\right)\right\}$$

Substitute into above,

$$\boxed{\sin\theta = \left(\frac{\cos\left[\sin^{-1}(h/R_{old})\right]}{\sqrt{1-(h/R_{new})^2}}\right)\frac{R_{old}}{R_{new}}}$$

exact

@

h

(★★)

skip

- **Approximate for φ small:**

$$\cos\varphi \simeq 1 - \frac{1}{2}\varphi^2 + \cdots \qquad (\varphi \ll 1)$$

$$\sin^2\varphi \simeq \varphi + \mathcal{O}(\varphi^3) \qquad \text{(in radians)}$$

$$\therefore \sin\theta \simeq \left(1 - \frac{1}{2}\varphi^2 + \cdots\right)\left(1 + \left(\frac{R_{old}}{R_{new}}\right)^2 \frac{1}{2}\varphi^2 + \cdots\right) \times \left(\frac{R_{old}}{R_{new}}\right)$$

$$\Rightarrow \sin\theta \simeq \left(1 - \frac{1}{2}\varphi^2 + \left(\frac{R_{old}}{R_{new}}\right)^2 \frac{1}{2}\varphi^2 + \cdots\right)\left(\frac{R_{old}}{R_{new}}\right)$$

$$= \frac{R_{old}}{R_{new}} - \left(1 - \left(\frac{R_{old}}{R_{new}}\right)^2\right)\left(\frac{R_{old}}{R_{new}}\right)\frac{1}{2}\varphi^2 + \cdots .$$

or,

$$\sin\theta \simeq \left[1 - \frac{1}{2}\varphi^2\left(1 - \left(\frac{R_{old}}{R_{new}}\right)^2\right) + \cdots\right]\frac{R_{old}}{R_{new}}$$

Recall: $\sin\varphi = \dfrac{h}{R_{old}}$

so if $h/R_{old} \ll 1$, then

$$\varphi \simeq \frac{h}{R_{old}} \qquad \text{(in radians)}$$

so that:

$\rightsquigarrow$

$$\sin\theta \simeq \left[1 - \frac{1}{2}\left(\frac{h}{R_{old}}\right)^2 \left(1 - \left(R_{old}/R_{new}\right)^2\right) + \cdots \right]\left(\frac{R_{old}}{R_{new}}\right)$$

When $h = 0$ above reduces to :

$$\sin\theta \simeq \frac{R_{old}}{R_{new}}$$

When is this a good approximation?

$\rightsquigarrow$ when $\ll$

Example :

$h \simeq$ 45 mm - 55 mm

$R_{old} \simeq$ 101.6 mm

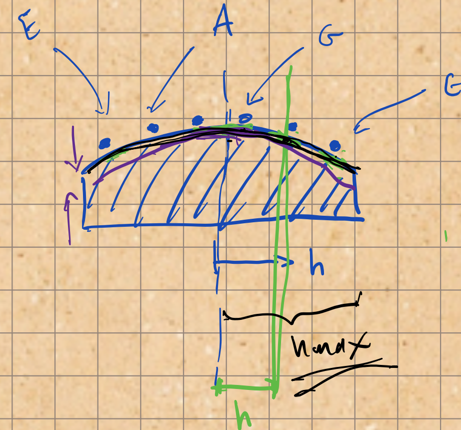
$R_{new} \simeq$ 304.8 mm

$= 0.13$

$= 1$

$\rightsquigarrow$ so $\sin\theta = \dfrac{R_{old}}{R_{new}}$ good approximation to within $\sim 13\%$ accuracy.

A Better Approximation



the problem

Equation (**) is valid at h.
So radius of curvature, R_{new}, (equivalently θ) is correct at h but less accurate elsewhere.
E.g., at h corresponding to E string, curvature will be accurate there, but less so around D or G string, etc. Or, if we set $h=0$ then formula accurate at D & G string but less so at E string.

a solution

One solution is to distribute error uniformly across entire range,

$$h \in [0, h_{max}]. \qquad (h_{max} \equiv w/2)^*$$

This can be done by averaging over this range.

$$\boxed{\sin\theta = \gamma(h) \longrightarrow \sin\theta = \int_0^{h_{max}} \frac{dh}{h_{max}}\, \gamma(h)}$$

* (w = width of neck.)

Applying this prescription to (**), i.e. to:

$$\sin\theta = \left(\frac{\cos[\arcsin(h/R_{old})]}{\sqrt{1-(h/R_{new})^2}} \right) \left(\frac{R_{old}}{R_{new}} \right) \quad \underline{\underline{\text{exact}}}$$

leads to the averaged version:

$$\sin\theta = \int_0^{h_{max}} \frac{dh}{h_{max}} \frac{\cos[\sin^{-1}(h/R_{old})]}{\sqrt{1-(h/R_{new})^2}} \frac{R_{old}}{R_{new}}$$

Solve integral using Mathematica:

$$\boxed{\sin\theta = \frac{R_{old}}{h_{max}} \times E\left(\sin^{-1}\left(\frac{h_{max}}{R_{new}}\right) \middle| \left(\frac{R_{new}}{R_{old}}\right)^2 \right)}$$

exact averaged version

where $E(\varphi|z)$ is elliptic integral of 2^{nd} kind.

If $\varphi = \sin^{-1}\left(\frac{h_{max}}{R_{new}}\right)$ is small,

$$E(\varphi|z) \simeq \varphi - \frac{z}{6}\varphi^3 + \cdots \quad \leftarrow \text{approximation}$$

so,

$$\sin\theta \simeq \frac{R_{old}}{h_{max}} \times \left[\sin^{-1}\left(\frac{h_{max}}{R_{new}}\right) - \frac{1}{6}\left(\frac{R_{new}}{R_{old}}\right)^2 \left(\sin^{-1}\frac{h_{max}}{R_{new}}\right)^3 + \dots \right]$$

$$\Rightarrow \quad \sin\theta \simeq \frac{R_{old}}{\cancel{h_{max}}} \frac{\cancel{h_{max}}}{R_{new}} - \frac{1}{6}\frac{R_{old}}{h_{max}}\left(\frac{R_{new}}{R_{old}}\right)^2 \left(\frac{h_{max}}{R_{new}}\right)^3$$

$$\Rightarrow \quad \sin\theta \simeq \frac{R_{old}}{R_{new}} - \frac{1}{6}\frac{h_{max}^2}{R_{old}\,R_{new}}.$$

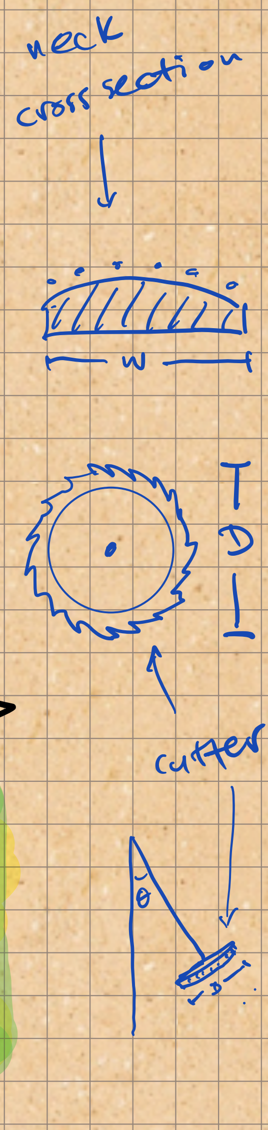
Equivalently,

$$\sin\theta \simeq \left(1 - \frac{1}{6}\left(\frac{h_{max}}{R_{old}}\right)^2\right)\frac{R_{old}}{R_{new}}$$

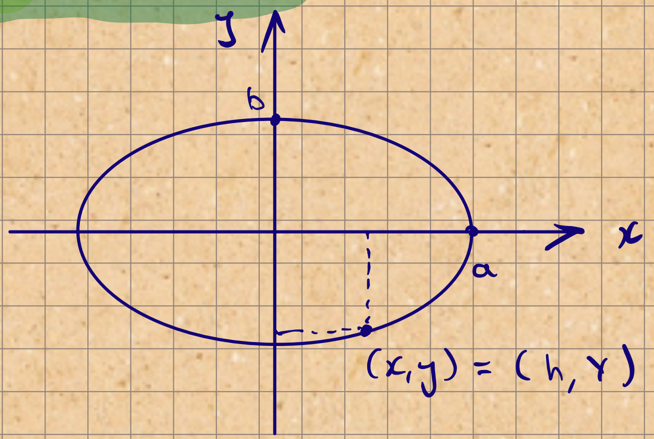
In terms of $\begin{cases} w \equiv 2h_{max} \\ R_{old} \equiv \dfrac{D}{2} \end{cases}$,

$$\sin\theta \simeq \left(1 - \frac{1}{6}\left(\frac{w/2}{D/2}\right)^2 + \dots\right)\frac{D/2}{R_{new}} \Rightarrow$$

$$\boxed{\sin\theta \simeq \left(1 - \frac{1}{6}\left(\frac{w}{D}\right)^2 + \dots\right)\frac{D}{2R_{new}}}$$



Again : think about ellipse more closely.



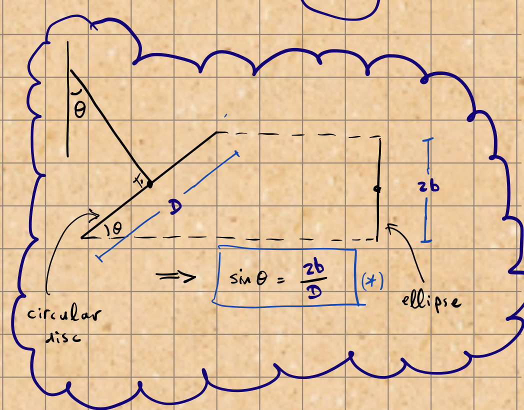
defining eqn for ellipse :

$$\boxed{\frac{y^2}{b^2} + \frac{x^2}{a^2} \equiv 1}$$

Quantities a & b determined in terms of D & θ,

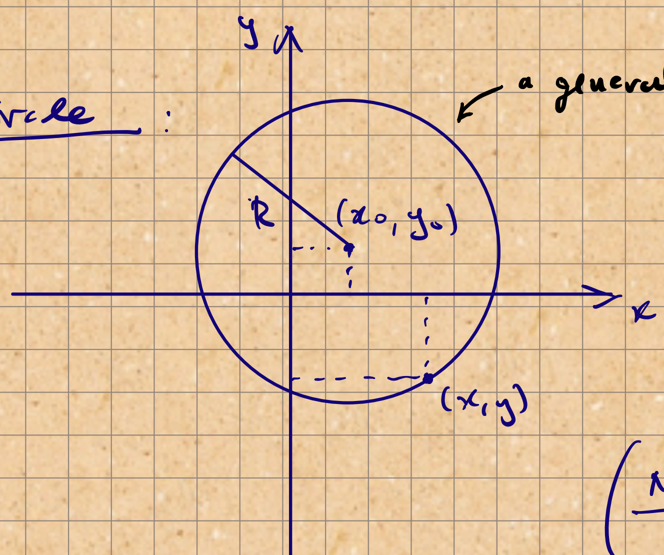
$$\boxed{a = \frac{D}{2}} \quad \& \quad \boxed{b = \frac{D}{2} \sin\theta}$$

(**check** : when $\theta = 90°$, $b = a = D/2$
so ellipse becomes a circle
as expected.)



⤳ Want to tune θ such that curvature close to that of circle of radius R_{new} within width of neck, $x \in [-\frac{w}{2}, \frac{w}{2}]$.

circle :



a general circle of radius R & centred at some (x_0, y_0). The equation is:

$$\boxed{(y-y_0)^2 + (x-x_0)^2 = R^2}$$

(**Note** : It need not be centred at $(x_0, y_0) = (0, y_0)$.)

The best fit circle at a point (x, y) of ellipse determined by 1^{st} & 2^{nd} derivative

ellipse:

$$\boxed{\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1}$$

(general equation for an ellipse centred at $(0,0)$)

$\Rightarrow^{*}$

$$y = -\frac{b}{a}(a^2 - x^2)^{1/2} \qquad \text{(A1)}$$

$$\frac{dy}{dx} = \frac{b}{a}\frac{x}{(a^2 - x^2)^{1/2}} \qquad \text{(A2)}$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\frac{a^2}{(a^2 - x^2)^{3/2}} \qquad \text{(A3)}$$

*(-ve root i.e. 3^{rd} & 4^{th} quadrant)

circle:

$$\boxed{(y - y_0)^2 + (x - x_0)^2 = R^2}$$

(general equation for a circle centred at (x_0, y_0))

$\Rightarrow$

$$y = -(R^2 - (x - x_0)^2)^{1/2} + y_0 \qquad \text{(B1)}$$

$$\frac{dy}{dx} = \frac{x - x_0}{(R^2 - (x - x_0)^2)^{1/2}} \qquad \text{(B2)}$$

$$\frac{d^2y}{dx^2} = \frac{R^2}{(R^2 - (x - x_0)^2)^{3/2}} \qquad \text{(B3)}$$

- The curvatures of ellipse & circle will be the same at a point $(x,y) = (h, y(h))$ if and only if 1^{st} & 2^{nd} derivatives of $y(x)$ in (A) & (B) are equal at $(h, y(h))$, i.e. (A2) = (B2) & (A3) = (B3) at $x = h$.

1^{st} deriv.: (A2) = (B2) @ $x = h$

$$\frac{b}{a}\frac{h}{(a^2-h^2)^{1/2}} \overset{!}{=} \frac{h-x_0}{(R^2-(h-x_0)^2)^{1/2}} \Rightarrow$$

$$\left(\frac{b}{a}\right)^2 \frac{h^2}{a^2-h^2}\left(R^2-(h-x_0)^2\right) = (h-x_0)^2 \Rightarrow$$

$$\left(\frac{b}{a}\right)^2 \frac{(hR)^2}{a^2-h^2} = (h-x_0)^2\left(1+\left(\frac{b}{a}\right)^2\frac{h^2}{a^2-h^2}\right) \Rightarrow$$

$$\left(\frac{b}{a}\right)^2 (hR)^2 = (h-x_0)^2\left(a^2-h^2+\left(\frac{b}{a}\right)^2 h^2\right)$$

$$= (h-x_0)^2\left(a^2-\left(1-(b/a)^2\right)h^2\right)$$

traditionally work in terms of eccentricity, e,

$$\boxed{e \equiv \sqrt{1-b^2/a^2}},$$

in terms of which,

$$(1-e^2)(hR)^2 = (h-x_0)^2(a^2-e^2h^2)$$

$$\Rightarrow (h-x_0)^2 = \frac{(1-e^2)(hR)^2}{a^2-e^2h^2}$$

δr_1

$$R^2 - (h-x_0)^2 = R^2 - \frac{(1-e^2)(hR)^2}{a^2 - e^2 h^2}$$

$$= \frac{(Ra)^2 - \cancel{(ehR)^2} - (hR)^2 + \cancel{(ehR)^2}}{a^2 - (eh)^2}$$

$$\boxed{R^2 - (h-x_0)^2 = \left(\frac{a^2 - h^2}{a^2 - (eh)^2}\right) R^2} \qquad (1)$$

2nd deriv.: (A3) = (B3) @ $x = h$

$$\left.\begin{aligned} \frac{b}{a} \frac{a^2}{(a^2 - h^2)^{3/2}} \stackrel{!}{=} \frac{R^2}{(R^2 - (h-x_0)^2)^{3/2}} \\ (1) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \sqrt{1-e^2} \, \frac{a^2}{\cancel{(a^2 - h^2)^{3/2}}} = R^2 \left(\frac{a^2 - (eh)^2}{\cancel{a^2 - h^2}}\right)^{\frac{3}{2}} \left(\frac{1}{R^2}\right)^{3/2}$$

$$\Rightarrow \sqrt{1-e^2} \, a^2 = \frac{1}{R} \left(a^2 - (eh)^2\right)^{3/2}$$

$$\Rightarrow \boxed{(1-e^2) a^4 R^2 = (a^2 - e^2 h^2)^3} \qquad (2)$$

Consistency checks: $\boxed{a = \frac{D}{2} \quad \& \quad b = \frac{D}{2}\sin\theta}$

$$\Rightarrow \quad e \equiv \sqrt{1-(b/a)^2}$$
$$= \sqrt{1-\sin^2\theta}$$
$$= \sqrt{\cos^2\theta}$$

$$\Rightarrow \quad \boxed{e = \cos\theta}$$

at $h = a$:
(or $h = D/2$)

$$(2) \Rightarrow \quad \cancel{(1-e^2)}\,a^{\cancel{4}}R^2 = (1-e^2)^{\cancel{3}\,2}\,a^{\cancel{6}\,2}$$

$$\Rightarrow \quad R^2 = (1-e^2)^2 a^2$$

$$\Rightarrow \quad \boxed{R = (1-e^2)\,a} \qquad \text{at } h = a$$

In terms of θ & D,

$$R = (1-\cos^2\theta)\,a$$
$$= \sin^2\theta \; \frac{D}{2} \quad \Rightarrow \quad \boxed{\begin{gathered}\sin\theta = \sqrt{\frac{D/2}{R}} \\ \text{at } h = D/2\end{gathered}}$$

at $h=0$:

$$(2) \Rightarrow \quad (1-e^2)\cancel{a^4}R^2 = a^{\cancel{6} \to 2}$$

$$\Rightarrow R^2 = \frac{a^2}{1-e^2}$$

$$\Rightarrow \boxed{R = \frac{a}{\sqrt{1-e^2}}}$$

In terms of θ, when $h=0$:

$$R = \frac{a}{\sqrt{1-\cos^2\theta}}$$

$$= \frac{a}{\sin\theta}$$

$$= \frac{D/2}{\sin\theta}$$

$$\Rightarrow \boxed{\sin\theta = \frac{D/2}{R} \quad \text{at } h=0}$$

$D \sim 4''$, $R \sim 12''$, $w \sim 2.4''$

from (2):

$$(1-e^2)\,a^4 R^2 = (a^2 - e^2 h^2)^3 \Rightarrow$$

$$\sin^2\theta \left(\frac{D}{2}\right)^4 R^2 = \left(\left(\frac{D}{2}\right)^2 - \cos^2\theta\, h^2\right)^3 \Rightarrow$$

$$\sin^2\theta \cancel{\left(\frac{D}{2}\right)^4} R^2 = \left(\frac{D}{2}\right)^{\cancel{6}\,2} \left(1 - \left(\frac{2h}{D}\right)^2 \cos^2\theta\right)^3 \Rightarrow$$

$$\boxed{\left(\frac{2R}{D}\right)^2 \sin^2\theta = \left(1 - \left(\frac{2h}{D}\right)^2 \cos^2\theta\right)^3} \longleftarrow \underline{\underline{\text{exact}}}$$

$$= 1 - 12\left(\frac{h}{D}\right)^2 \cos^2\theta + O\left((h/D)^4\right)$$

__approximate__ $\longrightarrow$

(for h/D small)

$$= 1 - 12\left(\frac{h}{D}\right)^2 + 12\left(\frac{h}{D}\right)^2 \sin^2\theta + \cdots$$

$$\Rightarrow \left[4\left(\frac{R}{D}\right)^2 - 4\times 3\left(\frac{h}{D}\right)^2\right] \sin^2\theta \simeq 1 - 12(h/D)^2 + \cdots$$

$$\Rightarrow \left(\frac{2R}{D}\right)^2 \left(1 - 3\left(\frac{h}{\cancel{D}}\right)^2 \left(\frac{\cancel{D}}{R}\right)^2\right) \sin^2\theta \simeq 1 - 12\left(\frac{h}{D}\right)^2 + \cdots$$

$$\Rightarrow \left(\frac{2R}{D}\right)^2 \sin^2\theta \simeq \frac{1 - 12(h/D)^2 + \cdots}{1 - 3(h/R)^2}$$

$$\Rightarrow \sin^2\theta \simeq \left(\frac{D}{2R}\right)^2 \left(1 - 12\left(\frac{h}{D}\right)^2 + 3\left(\frac{h}{R}\right)^2 + \cdots\right)$$

$$= \left(\frac{D}{2R}\right)^2 \left(1 - 3h^2\left(\left(\frac{2}{D}\right)^2 - \frac{1}{R^2}\right) + \cdots\right)$$

$$\Rightarrow \sin^2\theta \simeq \left(\frac{D}{2R}\right)^2 \left(1 - 12\left(\frac{h}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$\Rightarrow \boxed{\sin\theta \simeq \frac{D}{2R} \left(1 - 6\left(\frac{h}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)}$$

(assuming $h/D \ll 1$.)

Average over $h \in [0, w/2]$:

$$\langle \sin\theta \rangle = \int_0^{h_{max}} \frac{dh}{h_{max}} \frac{D}{2R} \left(1 - 6\left(\frac{h}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$= \frac{1}{h_{max}} \frac{D}{2R} \left(h_{max} - \frac{6}{D^2} \frac{h_{max}^3}{3} \left(1 - (D/2R)^2\right) + \cdots \right)$$

$$= \frac{D}{2R} \left(1 - \frac{2}{D^2} \left(\frac{w}{2}\right)^2 \left(1 - (D/2R)^2\right) + \cdots \right)$$

$$= \frac{D}{2R} \left(1 - \frac{1}{2} \left(\frac{w}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$\Rightarrow \boxed{\langle \sin\theta \rangle = \frac{D}{2R} \left\{ 1 - \frac{1}{2} \left[1 - \left(\frac{D}{2R}\right)^2 \right] \left(\frac{w}{D}\right)^2 + \cdots \right\}}$$

Spread in $\sin\theta$ (standard deviation): $\boxed{\sigma = \sqrt{\langle \sin^2\theta \rangle - \langle \sin\theta \rangle^2}}$

$$\langle \sin\theta \rangle^2 \simeq \left(\frac{D}{2R}\right)^2 \left\{ 1 - \left(1 - \left(\frac{D}{2R}\right)^2\right)\left(\frac{w}{D}\right)^2 + \cdots \right\}$$

$$\langle \sin^2\theta \rangle \simeq \int_0^{h_{max}} \frac{dh}{h_{max}} \left(\frac{D}{2R}\right)^2 \left(1 - 12\left(\frac{h}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$\simeq \left(\frac{D}{2R}\right)^2 \frac{1}{h_{max}} \left(h_{max} - \frac{12}{D^2} \frac{h_{max}^3}{3} \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$= \left(\frac{D}{2R}\right)^2 \left(1 - 4\left(\frac{h_{max}}{D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$= \left(\frac{D}{2R}\right)^2 \left(1 - 4\left(\frac{w}{2D}\right)^2 \left(1 - \left(\frac{D}{2R}\right)^2\right) + \cdots \right)$$

$$\Rightarrow \boxed{\sigma = 0}$$

$$s(h) \equiv 1 - Ah^2, \quad A = \frac{6}{D^2}\left(1 - \left(\frac{D}{2R}\right)^2\right)$$

$$\langle s \rangle = \int_0^{h_{max}} \frac{dh}{h_{max}} \left(1 - Ah^2\right)$$

$$= 1 - A \frac{h_{max}^2}{3}$$

$$\langle s^2 \rangle = \int_0^{h_{max}} \frac{dh}{h_{max}} \left(1 - Ah^2\right)^2$$

$$= 1 - \frac{2}{3} A h_{max}^2 + \frac{1}{5} A^2 h_{max}^4$$

$$\therefore \sigma_s = \sqrt{\langle s^2 \rangle - \langle s \rangle^2}$$

$$= \sqrt{\cancel{1} - \cancel{\frac{2}{3} A h_{max}^2} + \frac{1}{5} A^2 h_{max}^4 - \left(\cancel{1} + \frac{1}{9} A^2 h_{max}^4 - \cancel{\frac{2}{3} A h_{max}^2}\right)}$$

$$= \sqrt{\left(\frac{1}{5} - \frac{1}{9}\right) A^2 h_{max}^4}$$

$$= \sqrt{\frac{4}{45} \left(A h_{max}^2\right)^2}, \quad h_{max} = \frac{w}{2}$$

$$= \sqrt{\frac{4}{45} \left(A w^2/4\right)^2}$$

$$= \sqrt{\frac{4}{45} \frac{1}{16} \left(A w^2\right)^2}$$

$$= \sqrt{\frac{1}{4 \times 45}} \, A w^2$$

$$\Rightarrow \boxed{\sigma_s = \frac{1}{6\sqrt{5}} A w^2}$$

$$A = \frac{6}{D^2}\left(1 - \left(\frac{D}{2R}\right)^2\right)$$

$\Rightarrow$

$$\sigma_s = \frac{1}{\cancel{6}\sqrt{5}} \frac{\cancel{6}}{D^2}\left(1 - \left(\frac{D}{2R}\right)^2\right) w^2 \Rightarrow$$

$\times \frac{D}{2R}$

$$\boxed{\sigma_s = \frac{1}{\sqrt{5}} \frac{D}{2R}\left(1 - \left(\frac{D}{2R}\right)^2\right)\left(\frac{w}{D}\right)^2}$$

too big !!!

leads to 1-2° errors.

Include further higher order terms :

$$\boxed{\left(\frac{2R}{D}\right)^2 \sin^2\theta = \left(1 - \left(\frac{2h}{D}\right)^2 \cos^2\theta\right)^3}$$

$$\left(\begin{aligned} (1-x)^3 &= 1 - 3x + 3x^2 - x^3 \\ &\simeq 1 - 3x + 3x^2 + \cdots \end{aligned} \right)$$

$$\Rightarrow \underbrace{\left(\frac{2R}{D}\right)^2}_{\equiv \alpha^2} \sin^2\theta \simeq 1 - 3\left(\frac{2h}{D}\right)^2 \cos^2\theta + $$

$$+ 3\left(\frac{2h}{D}\right)^4 \cos^4\theta + \cdots$$

$$= 1 - 3z^2(1-\sin^2\theta) + 3z^4(1-\sin^2\theta)^2 + \cdots$$

$$\Rightarrow \alpha^2 \sin^2\theta \simeq 1 - 3z^2 + 3z^2\sin^2\theta +$$

$$+ 3z^4 + 3z^4 \sin^4\theta$$

$$- 6z^4 \sin^2\theta + \cdots$$

$$\Rightarrow 3z^4(\sin^2\theta)^2 + (3z^2 - 6z^4 - \alpha^2)(\sin^2\theta)$$

$$+ (1 - 3z^2 + 3z^4) = 0$$

$$\Rightarrow \sin^2\theta \simeq$$

$$\simeq \frac{\alpha^2 + 6z^4 - 3z^2 \pm \sqrt{(\alpha^2+6z^4-3z^2)^2 - 12z^4(3z^4-3z^2+1)}}{6z^4}$$

$$z = \frac{2h}{D} \quad , \quad \alpha \equiv \frac{2R}{D}$$

$$\left(\frac{2R}{D}\right)^2 \sin^2\theta = \left(1 - \left(\frac{2h}{D}\right)^2 \cos^2\theta\right)^3 \Rightarrow$$

$$\alpha^2 \sin^2\theta = (1 - z^2\cos^2\theta)^3$$

$$= (1 - z^2\cos^2\theta)(1 + z^4\cos^4\theta - 2z^2\cos^2\theta)$$

$$\Rightarrow \alpha^2(1-\cos^2\theta) = (1 - z^2\cos^2\theta)(1 + (z^2\cos^2\theta)^2 - 2z^2\cos^2\theta)$$

$$\Rightarrow \boxed{(z^6 - 3z^4 + 3z^2 - 1) + (\alpha^2 - 3z^6 + 6z^4 - 3z^2)(\sin^2\theta) + (3z^6 - 3z^4)(\sin^2\theta)^2 + (-z^6)(\sin^2\theta)^3 = 0}$$

$$\boxed{\alpha > z > 0}$$

$$(1-e^2)\,a^4R^2 = (a^2 - e^2h^2)^3 \quad \Rightarrow$$

$$\boxed{R(h) = \frac{(a^2 - e^2h^2)^{3/2}}{a^2(1-e^2)^{1/2}}} \qquad \underline{\underline{\text{exact}}}.$$

$$\langle R \rangle \equiv \int_0^{h_{max}} \frac{dh}{h_{max}}\, R(h)$$

$$= \int_0^{h_{max}} \frac{dh}{h_{max}} \frac{(a^2-e^2h^2)^{3/2}}{a^2(1-e^2)^{1/2}}$$

$$= \Bigg[\frac{(a^2-e^2h^2)^{1/2}}{a^2(1-e^2)^{1/2}} \frac{1}{8}\left(5a^2h - 2e^2h^3\right)$$

$$+ \frac{3a^{\cancel{4}\,2}/(8e)}{\cancel{a^2}(1-e^2)^{1/2}} \arctan\left(\frac{eh}{\sqrt{a^2-e^2h^2}}\right)\Bigg]_0^{h_{max}} \frac{1}{h_{max}}$$

$$\left.\begin{aligned}
\Rightarrow \langle R \rangle &= \frac{(a^2-e^2h_{max}^2)^{1/2}}{a^2(1-e^2)^{1/2}} \frac{1}{8}\left(5a^2 - 2e^2h_{max}^2\right) \\
&\quad + \frac{3a^2}{8e\,h_{max}} \frac{1}{(1-e^2)^{1/2}} \tan^{-1}\left(\frac{e\,h_{max}}{\sqrt{a^2-e^2h_{max}^2}}\right) \\
&a = \frac{D}{2},\ e = \cos\theta \ \ \& \ \ h_{max} = \frac{w}{2}
\end{aligned}\right\} \Rightarrow$$

$$\langle R\rangle = \frac{(D^2 - e^2 w^2)^{1/2}}{\sqrt{2}\,(1-e^2)^{1/2}}\,\frac{5}{8}\left(1 - \frac{2}{5}e^2\left(\frac{w}{D}\right)^2\right)$$

$$+ \frac{3}{8}\,\frac{(D/2)^2}{e(w/2)}\,\frac{1}{(1-e^2)^{1/2}}\tan^{-1}\left(\frac{ew/\cancel{2}}{\frac{D}{\cancel{2}}\sqrt{1-e^2(w/D)^2}}\right)$$

$$= \frac{5D}{8\sqrt{2}}\left(\frac{1-e^2(w/D)^2}{1-e^2}\right)^{1/2}\left(1 - \frac{2}{5}e^2\left(\frac{w}{D}\right)^2\right)$$

$$+ \frac{3D}{16}\left(\frac{D}{ew}\right)\frac{1}{(1-e^2)^{1/2}}\tan^{-1}\left(\frac{(ew/D)}{\sqrt{1-(ew/D)^2}}\right)$$

Natural parameter: $\boxed{\alpha \equiv \frac{w}{D}}$

$$\frac{\langle R\rangle}{D} = \frac{5}{8\sqrt{2}}\left(\frac{1-(e\alpha)^2}{1-e^2}\right)^{1/2}\left(1 - \frac{2}{5}(e\alpha)^2\right)$$

$$+ \frac{3}{16}\,\frac{1}{e\alpha}\,\frac{1}{(1-e^2)^{1/2}}\tan^{-1}\left(\frac{e\alpha}{\sqrt{1-(e\alpha)^2}}\right)$$

$$\boxed{\begin{aligned} 1-e^2 &= \sin^2\theta \\ e\alpha = \frac{ew}{D} &\equiv \beta \end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\langle R\rangle}{D} &= \frac{5}{8\sqrt{2}}\,\frac{\sqrt{1-\beta^2}}{\sin\theta}\left(1 - \frac{2}{5}\beta^2\right) \\ &+ \frac{3}{16}\,\frac{1}{\beta\sin\theta}\tan^{-1}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)\end{aligned}}$$

If $\beta \ll 1$

$$\left(\langle R\rangle \frac{2\sin\theta}{D}\right)^2 \simeq \left(\frac{59}{64} + \frac{15}{16\sqrt{2}}\right) - \left(\frac{87}{64} + \frac{11}{16\sqrt{2}}\right)\beta^2 + \cdots$$

$$\langle R^2\rangle \equiv \int_0^{h_{max}} \frac{dh}{h_{max}}\, R(h)^2$$

$$= \int_0^{h_{max}} \frac{dh}{h_{max}} \frac{(a^2 - e^2 h^2)^3}{a^4(1-e^2)}$$

$$= \frac{a^2}{1-e^2}\left(1 - e^2\left(\frac{h_{max}}{a}\right)^2 + \frac{3}{5}\left(\frac{e h_{max}}{a}\right)^4 - \frac{1}{7}\left(\frac{e h_{max}}{a}\right)^6\right)$$

$$= \frac{(D/2)^2}{1-e^2}\left(1 - e^2\left(\frac{w}{D}\right)^2 + \frac{3}{5}\left(\frac{ew}{D}\right)^4 - \frac{1}{7}\left(\frac{ew}{D}\right)^6\right)$$

but $\beta \equiv \frac{ew}{D}$ & $1-e^2 = \sin^2\theta$,

$$\frac{\langle R^2\rangle}{D^2} = \frac{1}{4}\frac{1}{\sin^2\theta}\left(1 - \beta^2 + \frac{3}{5}\beta^4 - \frac{1}{7}\beta^6\right) \Rightarrow$$

$$\boxed{\langle R^2\rangle\left(\frac{2\sin\theta}{D}\right)^2 = \left(1 - \beta^2 + \frac{3}{5}\beta^4 - \frac{1}{7}\beta^6\right)}$$

Therefore,

$$\sigma_R \equiv \sqrt{\langle R^2\rangle - \langle R\rangle^2}$$

$$\simeq \left(\frac{D}{2\sin\theta}\right)\sqrt{1-\beta^2+\cdots-\left(\frac{59}{64}+\frac{15}{16\sqrt{2}}\right)-\left(\frac{87}{64}+\frac{11}{16\sqrt{2}}\right)\beta^2}$$

$$\simeq \frac{D}{2\sin\theta}$$

$\sqrt{-1}$!! ?

at $\beta \to 0$

$$\langle R\rangle^2 = \left(\frac{D}{2\sin\theta}\right)^2\left[\frac{5}{\cancel{8}_{4}\sqrt{2}}\,\frac{\sqrt{1-\beta^2}}{\cancel{\sin\theta}}\left(1-\frac{2}{5}\beta^2\right) + \frac{3}{\cancel{16}_{8}}\,\frac{1}{\beta\,\cancel{\sin\theta}}\tan^{-1}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)\right]^2$$

Compute σ_R is limit $\beta \ll 1$:

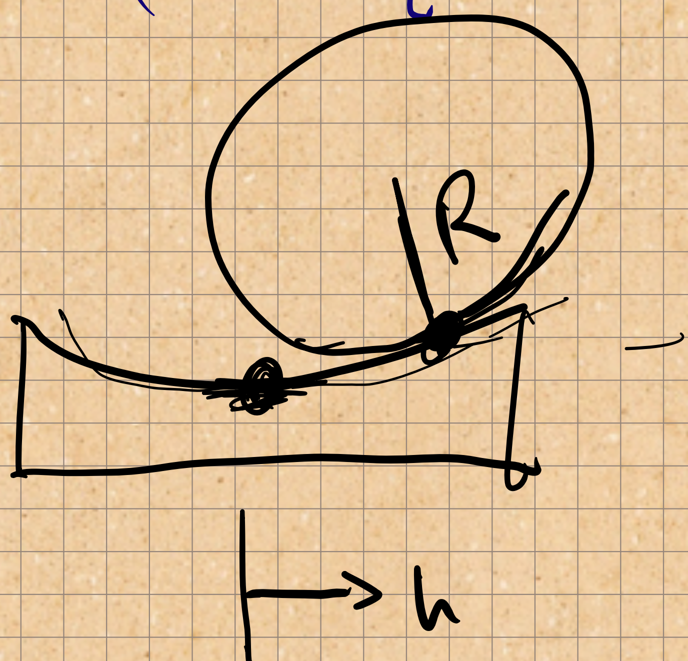
$$\sigma_R = \sqrt{\langle R^2 \rangle - \langle R \rangle^2}$$

$$= \left(\frac{D}{2\sin\theta}\right)\left\{\left(1-\beta^2+\frac{3}{5}\beta^4-\frac{1}{7}\beta^6\right)-\left[\frac{5}{\sqrt{2}}\sqrt{1-\beta^2}\left(1-\frac{2}{3}\beta^2\right)\right.\right.$$
$$\left.\left.+\frac{3}{8}\frac{1}{\beta}\tan^{-1}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right)\right]^2\right\}^{1/2}$$

But :

$$\frac{\langle R \rangle}{D} = \frac{1}{2\sin\theta}\left\{\frac{3+5\sqrt{2}}{8}+\frac{3(1-3\sqrt{2})}{16}\beta^2+\cdots\right\}$$

$$\Rightarrow \langle R \rangle^2 \simeq \left(\frac{D}{2\sin\theta}\right)^2\left(\frac{3+5\sqrt{2}}{8}\right)^2\left\{1+3\frac{(1-3\sqrt{2})}{3+5\sqrt{2}}\beta^2+\cdots\right\}$$



$$R(\theta) = \frac{D}{2\sin\theta}$$

$$R(h) = \frac{(a^2 - e^2 h^2)^{3/2}}{a^2 (1-e^2)^{1/2}} \Rightarrow$$

$$R(h) = \frac{a\,(1-(eh/a)^2)^{3/2}}{\sin\theta}$$

(1) $$\boxed{R(h) = \frac{D}{2\sin\theta}\left(1-\left(\frac{2\cos\theta}{D}\right)^2 h^2\right)^{3/2}}$$ exact.

$$\langle R\rangle = \frac{D}{2\sin\theta}\int_0^{h_{max}} \frac{dh}{h_{max}}\,(1-A^2h^2)^{3/2}, \quad \left(A = \frac{2\cos\theta}{D}\right)$$

$$= \frac{D}{2\sin\theta}\left[(1-A^2h_{max}^2)^{1/2}\left(\frac{5}{8}-\frac{1}{4}(Ah_{max})^2\right)+\frac{3}{8}\frac{\sin^{-1}(Ah_{max})}{Ah_{max}}\right]$$

(2) $$\boxed{\langle R\rangle = \frac{D}{2\sin\theta}\left[\left(1-\left(\cos\theta\,\frac{w}{D}\right)^2\right)^{1/2}\left(\frac{5}{8}-\frac{1}{4}\left(\cos\theta\,\frac{w}{D}\right)^2\right) + \frac{3}{8}\,\frac{\sin^{-1}(\cos\theta\; w/D)}{\cos\theta\; w/D}\right]}$$ exact

Also,

$$\langle R^2\rangle = \left(\frac{D}{2\sin\theta}\right)^2\int_0^{h_{max}}\frac{dh}{h_{max}}\,(1-A^2h^2)^3$$

$$= \left(\frac{D}{2\sin\theta}\right)^2\left[1-(Ah_{max})^2+\frac{3}{5}(Ah_{max})^4-\frac{1}{7}(Ah_{max})^6\right]$$

$$\langle R^2\rangle = \left(\frac{D}{2\sin\theta}\right)^2\left[1-\underbrace{(\cos\theta\; w/D)^2}_{\beta^2}+\frac{3}{5}(\cos\theta\; w/D)^4-\frac{1}{7}(\cos\theta\; w/D)^6\right]$$

$$\therefore\ \sigma = \sqrt{\langle R^2\rangle - \langle R\rangle^2}$$

$$= \left(\frac{D}{2\sin\theta}\right)\sqrt{\left(1-\beta^2+\ldots\right)-\left(\frac{49}{64}-\frac{41}{64}\beta^2+\ldots\right)}$$

$$\Rightarrow \delta \simeq \frac{D}{2\sin\theta}\sqrt{\frac{15}{64}}\left(1-\frac{64}{15}\left(1-\frac{41}{64}\right)\beta^2+\cdots\right)^{1/2}$$

$$\simeq \frac{D}{2\sin\theta}\sqrt{\frac{15}{64}}\left(1-\frac{32}{15}\,\frac{23}{64}\,\beta^2+\cdots\right)$$

$$\Rightarrow \boxed{\delta_R \simeq \frac{D}{2\sin\theta}\sqrt{\frac{15}{64}}\left(1-\frac{23}{30}\left(\cos\theta\,\frac{w}{D}\right)^2+\cdots\right)}$$

so, e.g. if $\begin{cases} \sin\theta \simeq 0.16 \\ D \simeq 4'' \end{cases}$

$\rightsquigarrow \boxed{\delta_R \sim 6.5''}$!!! <u>huge</u>

$$\sin\theta = \frac{D}{2R}\left(1-\left(\frac{2h}{D}\right)^2\left(1-\sin^2\theta\right)\right)^{3/2}$$

$$\boxed{R(h) = \frac{D}{2\sin\theta}\left(1-\left(\frac{2\cos\theta}{D}\right)^2 h^2\right)^{3/2}}$$ exact.

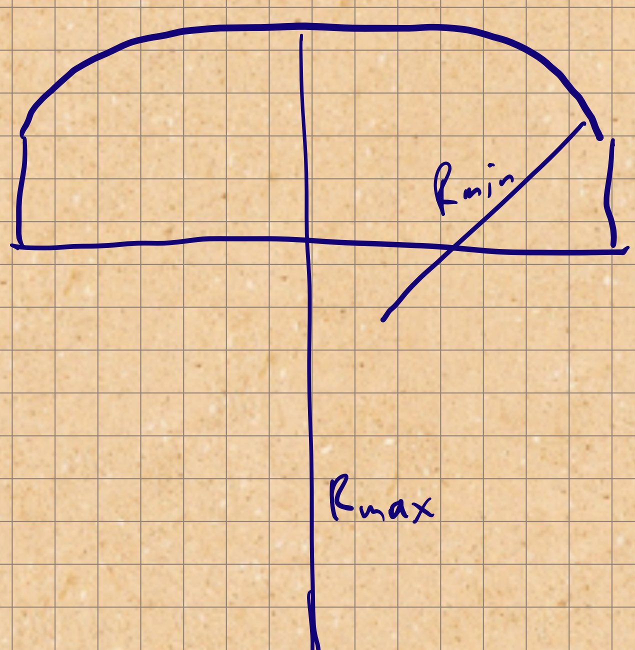
$$R_{max} - R_{min} = \frac{D}{2\sin\theta} - \frac{D}{2\sin\theta}\left(1-\left(\frac{w}{D}\right)^2\left(1-\sin^2\theta\right)\right)^{3/2}$$

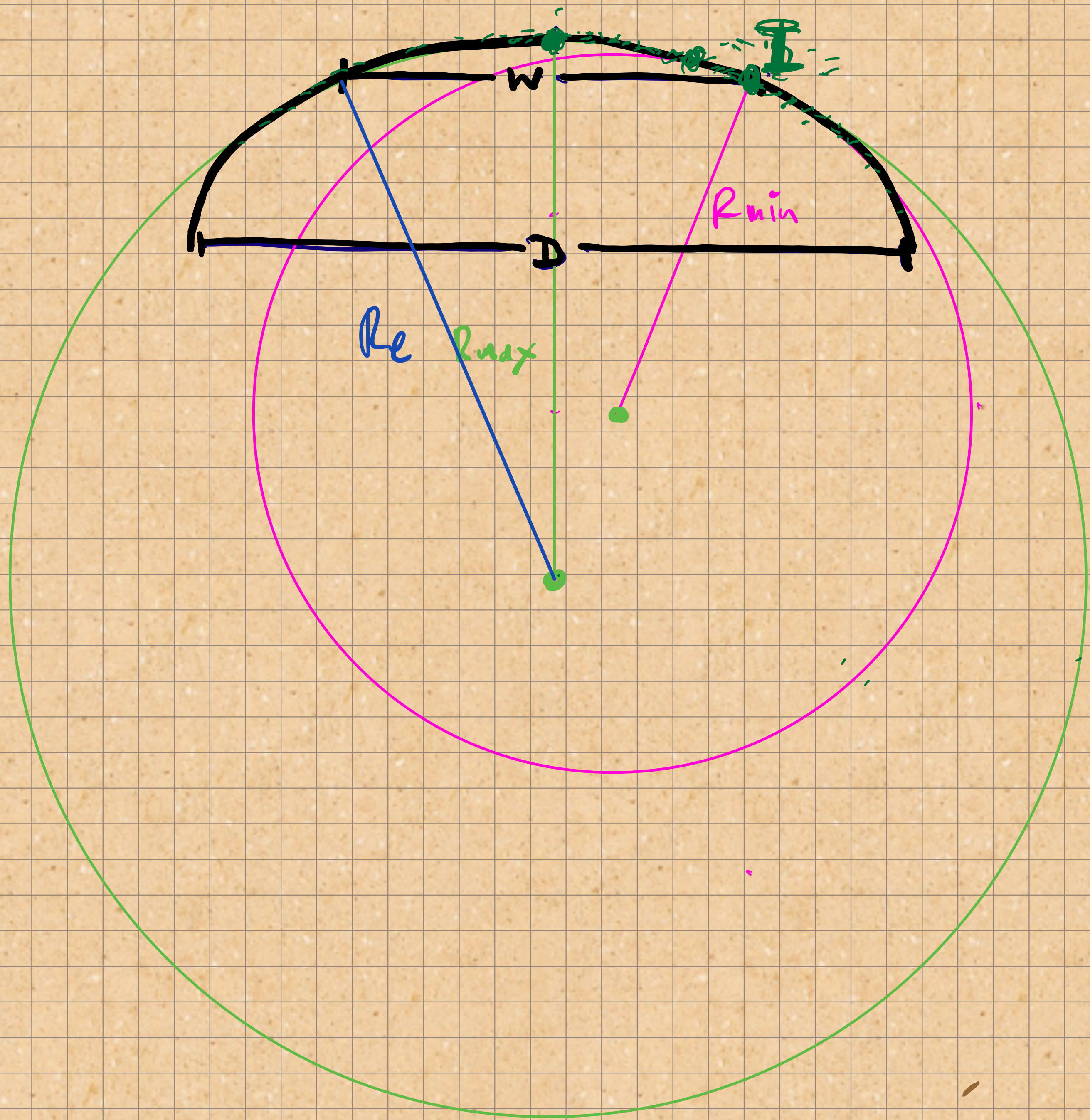
if: $\begin{cases} D = 4'' \quad \& \quad R = 12'' \\ w = 2'' \\ \sin\theta \simeq \dfrac{D}{2\times R} = \dfrac{4}{2\times 12} = \dfrac{1}{6} \end{cases}$

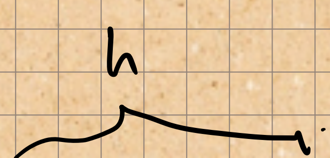
Then:
$$R_{max} - R_{min} = \frac{4}{2\times 1/6} - \frac{4}{2\times 1/6}\left(1-\left(\frac{2}{4}\right)^2\left(1-1/36\right)\right)^{3/2}$$

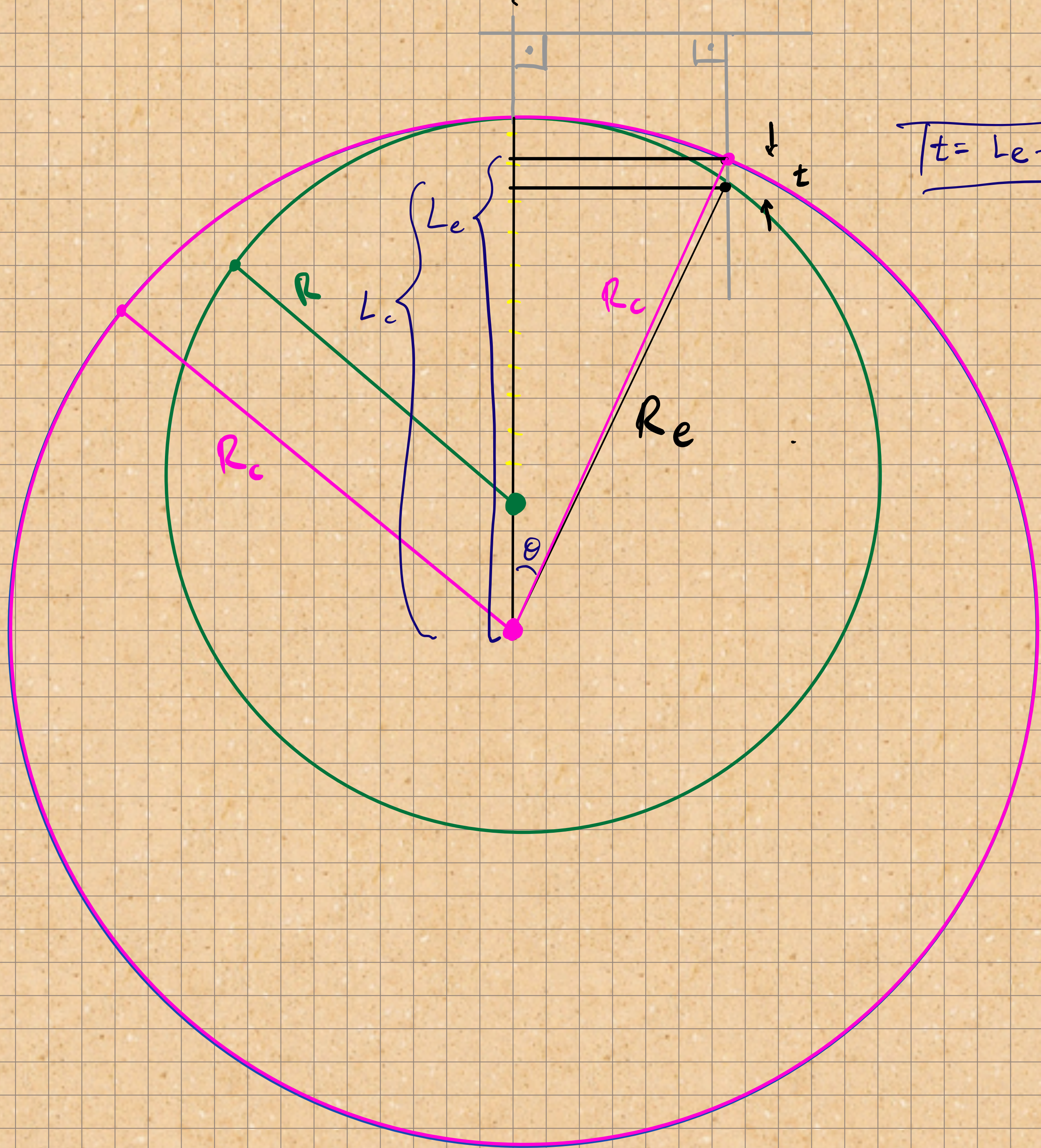
$$\simeq 12 - 12\left(1-\frac{35}{4\times 36}\right)^{3/2}$$

$$\boxed{R_{max} - R_{min} = 4''}$$




W
D
I
Re
Rmax
Rmin


h



$$\boxed{t = L_e - L_c}$$

$$\left.\begin{aligned} L_e &= R_e \cos\theta \\ L_c &\simeq R_c \cos\theta \\ R_e \sin\theta &= \frac{w}{2} \end{aligned}\right\} \Rightarrow \begin{aligned} t &= (R_e - R_c)\cos\theta \\ &= (R_e - R_c)\cos\left[\sin^{-1}\left(\frac{w}{2R_e}\right)\right] \end{aligned}$$

$$R_e^2 + (R_c - R)^2 - 2R_e(R_c - R)\cos\theta = R^2$$

$$\Rightarrow R_e = \frac{\cancel{2}(R_c - R)\cos\theta \pm \sqrt{\cancel{4}(R_c-R)^2\cos^2\theta - \cancel{4}\left[(R_c-R)^2 - R^2\right]}}{\cancel{2}}$$

$$R_e = (R_c - R)\cos\theta \pm \sqrt{(R_c-R)^2\cos^2\theta - (R_c^2 - 2R_cR)}$$

$\therefore$

$$R_e = (R_c - R)\cos\theta \pm \sqrt{(R_c-R)^2(-\sin^2\theta) + R^2}$$

$$= (R_c - R)\cos\theta \pm R\sqrt{1 - \left(\frac{R_c - R}{R}\right)^2 \sin^2\theta}$$

if $\theta = 0$ $\quad R_e \to R_c$ $\quad$ so +ve root

$$\boxed{R_e = (R_c - R)\cos\theta + R\sqrt{1 - \left(\frac{R_c - R}{R}\right)^2 \sin^2\theta}}$$

$$\left.\begin{aligned} x^2 + y_c^2 &= R_c^2 \\ x^2 + (y - y_0)^2 &= R^2 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} y_c &= \sqrt{R_c^2 - x^2} \\ y &= \sqrt{R^2 - x^2} + y_0 \\ \text{At } x = 0 \quad & y_c(0) = y(0) \end{aligned}\right\} \Rightarrow$$

$$R_c = R + y_0 \Rightarrow y_0 = R_c - R$$

Therefore,

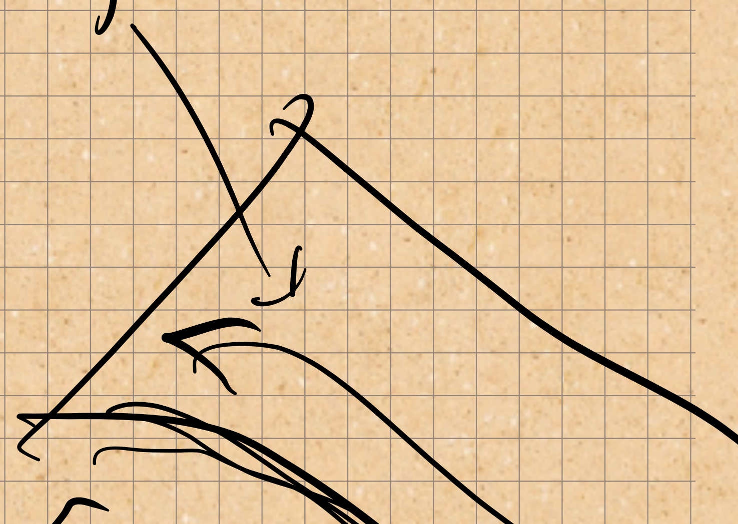
$$\left.\begin{cases} y_c = \sqrt{R_c^2 - x^2} \\ y = \sqrt{R^2 - x^2} + R_c - R \end{cases} \quad t \equiv y_c - y \right\} \Rightarrow$$

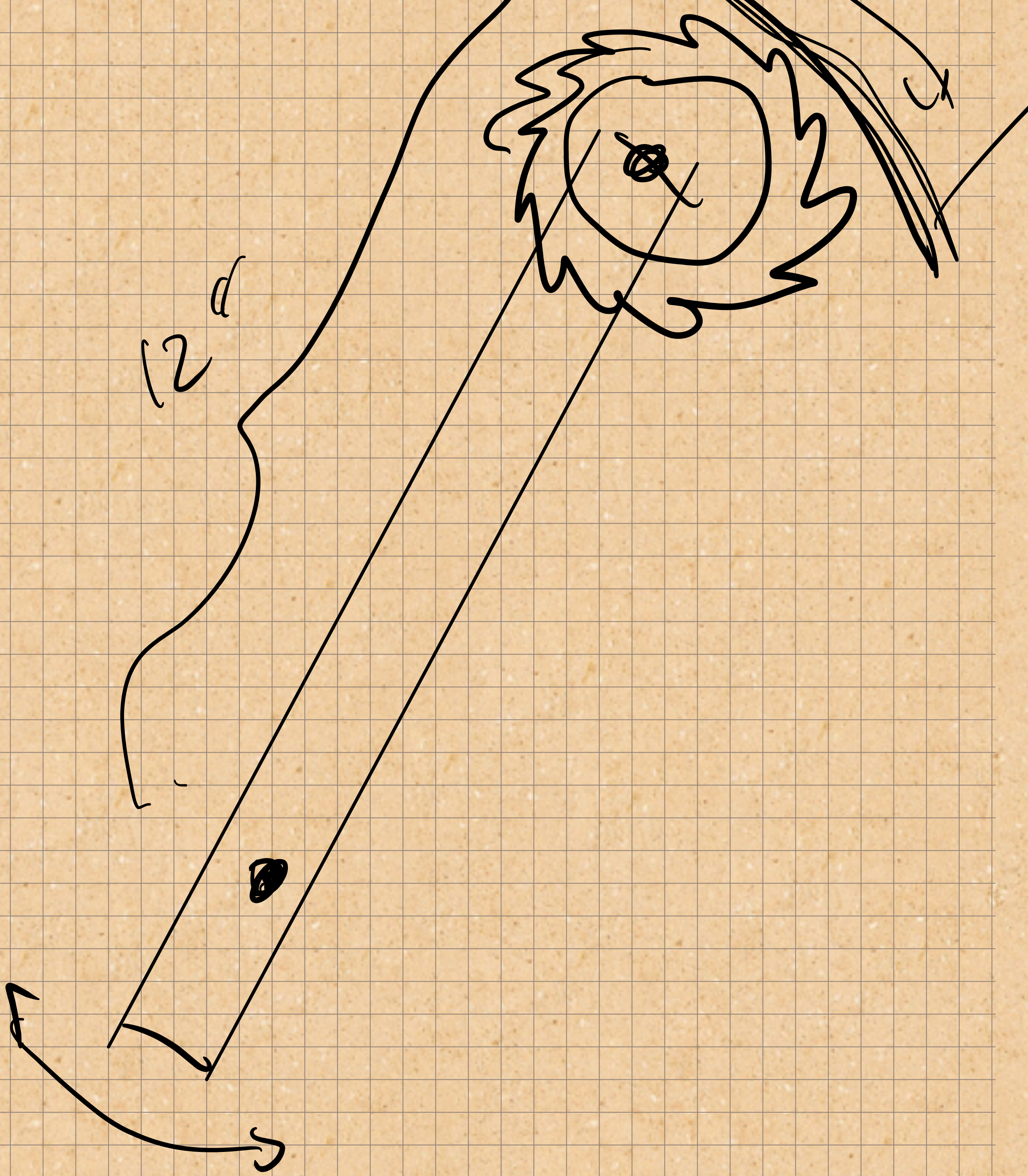
$$t = \sqrt{R_c^2 - h^2} - \sqrt{R^2 - h^2} - R_c + R$$

$$h = w/2$$

$$\boxed{t = \sqrt{R_c^2 - (w/2)^2} - \sqrt{R^2 - (w/2)^2} - (R_c - R)}$$

$\xi_i \geq 0$




12d

$R = R_{new}$ $\quad r = D/2 = R_{old}$

$w = h$

$\cancel{R = D/2}$

$$\sin\theta = \frac{R_{new} - (R_{new}^2 - w^2)^{1/2}}{r - (r^2 - w^2)^{1/2}}$$

$$\langle R \rangle = \frac{D}{2\sin\theta}\left[\left(1-\left(\cos\theta\,\frac{w}{D}\right)^2\right)^{1/2}\left(\frac{5}{8}-\frac{1}{4}\left(\cos\theta\,\frac{w}{D}\right)^2\right) + \frac{3}{8}\,\frac{\sin^{-1}(\cos\theta\; w/D)}{\cos\theta\;\; w/D}\right]$$

$$\sin\theta = \frac{D}{2\,R}\left[\left(1-(1-\sin^2\theta)\left(\frac{w}{D}\right)^2\right)^{1/2} \times\left(\frac{5}{8}-\frac{1}{4}(1-\sin^2\theta)\left(\frac{w}{D}\right)^2\right) + \frac{3}{8}\,\frac{\operatorname{asin}\left(\sqrt{1-\sin^2\theta}\;\; w/D\right)}{\sqrt{1-\sin^2\theta}\;\;\; w/D}\right]$$

<u>initial condition</u>:

$$y_0 = \frac{D}{2R}$$

for $i = 0, \ldots, 10$

$$y_{i+1} = \frac{D}{2R}\left[\left(1-(1-y_i^2)(w/D)^2\right)^{1/2} \times\left(\frac{5}{8}-\frac{1}{4}(1-y_i^2)(w/D)^2\right) + \frac{3}{8}\,\frac{\operatorname{asin}\left(\sqrt{1-y_i^2}\;\; w/D\right)}{\sqrt{1-y_i^2}\;\;\; w/D}\right]$$

$R_{old} \times 2 = D$

$R_{new} = R$

$y_0 = \alpha$

for $i = 0, 1, 2$

$$y_{i+1} = f(y_i)$$

end

print y_4

$\underline{i=0}$ $\quad y_1 = f(y_0)$

$\underline{i=1}$ $\quad y_2 = f(y_1)$

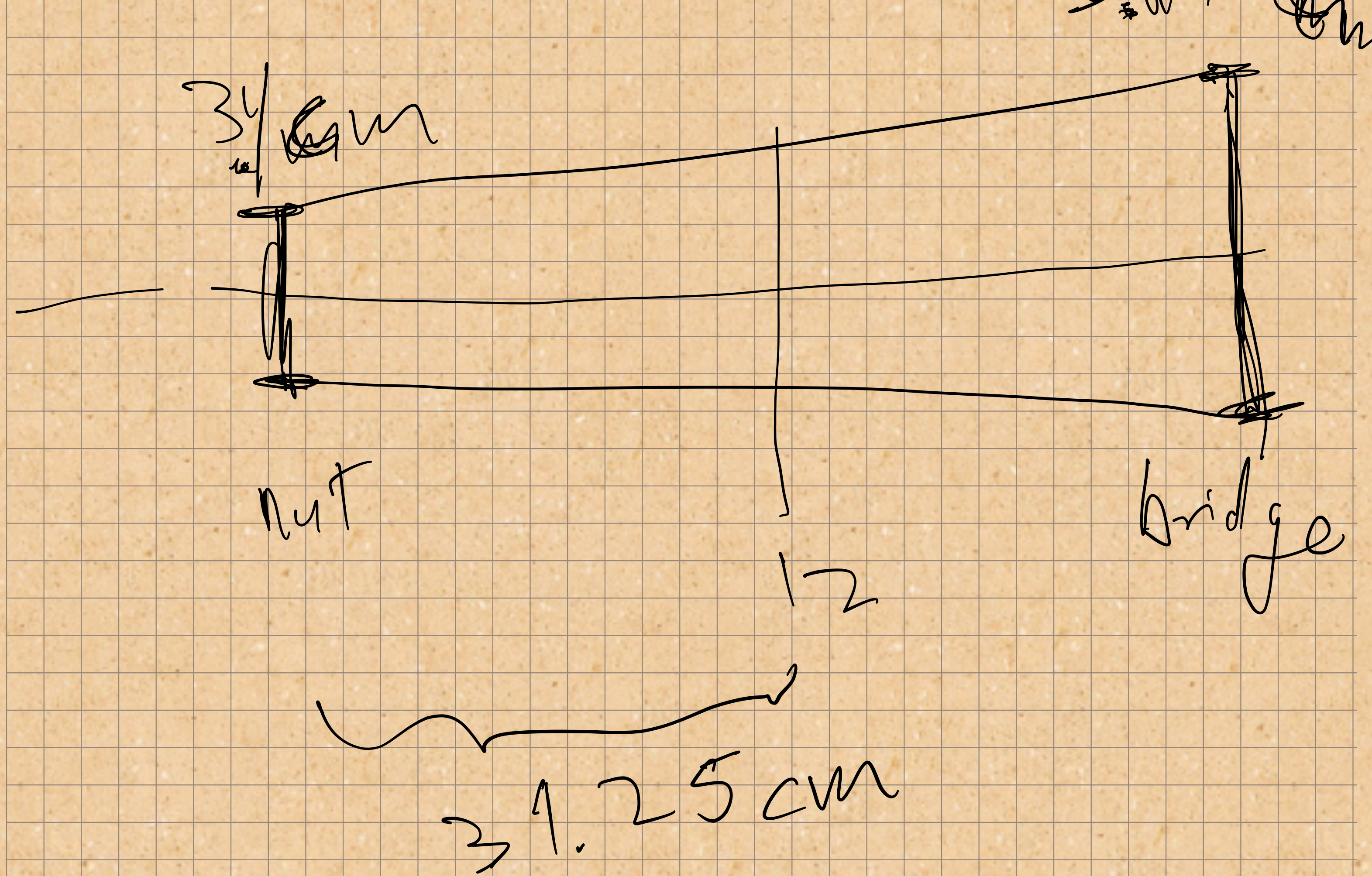
$\underline{i=2}$ $\quad y_3 = f(y_2)$

$$\boxed{y_3 = f(f(f(y_0)))}$$

$$\sigma = \sqrt{\langle R^2 \rangle - \langle R \rangle^2}$$

$$\langle R \rangle = 12''$$

$$\sigma = \ ?$$



$$R_D = R\left(1 + \frac{D}{X}\right)$$

$$y = ax + b$$

$$= \left(\frac{(5.1375 - 3.4)\,\text{cm}}{2 \times 31.25}\right)x + b$$

$$y(0) = \frac{3.4}{2}$$

$y = 0 \Rightarrow$

$$\left(\frac{(5.175 - 3.4)}{4 \times 31.25}\right) x_0 + \frac{3.4}{2} = 0 \Rightarrow$$

$$\Rightarrow x_0 = -\frac{4 \times 31.25 \times 3.4}{2 \times (5.175 - 3.4)} \text{ cm}$$

$$X = 120 \text{ cm}$$

$$R_d = \frac{R_n (X + D)}{X}$$

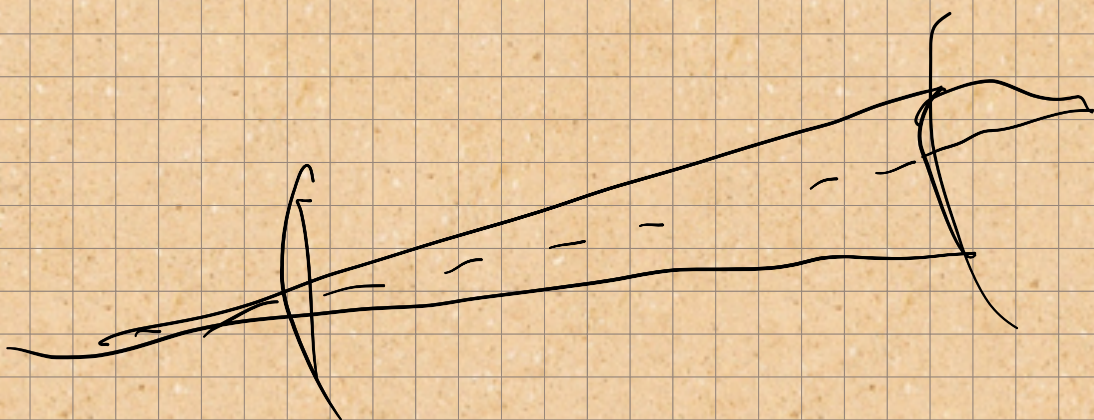
$$= \frac{12'' (120 + 31.25)}{120}$$

$= 15''$

$$R_{Bridge} = \frac{12''(120 + 2 \times 31.25)}{120}$$

$= 18''$

$7.5'' - 20''$



7.25, 9.5, 10, 12

14, 15, 16, 17, 20